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THE STABILITY OF THE PSEUDO-REGULAR PRECESSION OF A GYROSTAT†

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The perturbed motion of a gyrostat, which is very similar to regular Euler precession, is analysed. The perturbations considered include small displacements of the centre of gravity, the inclusion of rotors, and the loss of dynamic symmetry. Using the Kolmogorov–Arnol'd theory, the stability of conditionally periodic motion close to regular precession is established. © 1997 Elsevier Science Ltd. All rights reserved.

Consider the perturbed motion of a gyrostat in a uniform gravitational field. The unperturbed motion is taken as Euler inertial rotation about a fixed point of a solid with dynamic axial symmetry ($A = B$). The perturbations are caused by symmetric rotors, displacements of the centre of gravity of the body about a fixed point, and the loss of dynamic axial symmetry.

The analysis was carried out in canonical Poincaré variables L, ρ_1, ρ_2, λ, ω_1, ω_2, which are associated with the canonical Andoyer–Deprit variables L, G, $\mathcal{H}$, l, g, h by the relations [1]

$$L = L,\quad \rho_1 = L - G,\quad \rho_2 = G - \mathcal{H},\quad \lambda = l + g + h,\quad \omega_1 = -(g + h),\quad \omega_2 = -h$$

Then the Hamiltonian which describes the perturbed motion of the gyrostat will have the form

$$H = H_0 + \varepsilon(H_1 + H_2 + H_3),\quad H_0 = \frac{\rho_1(\rho_1 - 2L)}{2A} + \frac{L^2}{2C},\quad \varepsilon = \frac{A - B}{A}$$

Here ε is a small parameter (A and B are the moments of inertia of the gyrostat), H_0 is the unperturbed Hamiltonian (C is the axial moment of inertia), and the components of the perturbation function H_1, H_2, H_3 can be represented as

$$H_1 = \frac{\rho_1(\rho_1 - 2L)}{2B}\cos^2(\lambda + \omega_1)$$

$$H_2 = D\left[\frac{\alpha_1}{A}(\rho_1(\rho_1 - 2L))^{1/2}\sin(\lambda + \omega_1) + \frac{\alpha_2}{B}(\rho_1(\rho_1 - 2L))^{1/2}\cos(\lambda + \omega_1) + \frac{\alpha_3}{C}L\right]\Omega \tag{1}$$

$$H_3 = -mg\left\{\beta_1\left[b\sin(\lambda + \omega_1) + \frac{b_1}{2}((b_4 + 1)\sin(\lambda + \omega_2) + (b_4 - 1)\sin(\lambda + 2\omega_1 - \omega_2))\right] + \right.$$
$$+\beta_2\left[b\cos(\lambda + \omega_1) + \frac{b_1}{2}((b_4 + 1)\cos(\lambda + \omega_2) + (b_4 - 1)\cos(\lambda + 2\omega_1 - \omega_2))\right] +$$
$$\left. + \beta_3(b_2 + b_3\cos(\omega_2 - \omega_1))\right\}$$

where α_1, α_2, α_3, are the direction cosines of the axis of dynamic symmetry of the rotor about the associated axes, Ω is the relative angular velocity of the rotor (Ω = const) and J is its moment of inertia. We have also used the notation

$$b = \frac{(L - \rho_1 - \rho_2)(\rho_1(\rho_2 - 2L))^{1/2}}{(L - \rho_1)^2},\quad b_1 = \frac{(\rho_2(2L - 2\rho_1 - \rho_2))^{1/2}}{L - \rho_1},\quad b_2 = \frac{(L - \rho_1 - \rho_2)L}{(L - \rho_1)^2}$$

$$b_3 = \frac{(\rho_1\rho_2(\rho_1 - 2L)(2L - 2\rho_1 - \rho_2))^{1/2}}{(L - \rho_1)^2},\quad b_4 = \frac{L}{L - \rho_1}$$

$$\varepsilon D = J,\quad \beta_1\varepsilon = x_C,\quad \beta_2\varepsilon = y_C,\quad \beta_3\varepsilon = z_C$$

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(x_C, y_C, z_C are the coordinates of the centre of masses of the gyrostat).

It is clear from (1) that the component εH_1 of the perturbation function describes perturbations caused by the loss of dynamic axial symmetry of the gyrostat, εH_2 the perturbations due to the presence of rotors, and εH_3 the perturbations due to displacement of the centre of masses of the gyrostat relative to the fixed point.

When $\varepsilon = 0$ the general solution of the system of canonical equations has the form

$$L = L_0 = \text{const}, \quad \rho_1 = \rho_{10} = \text{const}, \quad \rho_2 = \rho_{20} = \text{const}$$
$$\lambda = n_1 t + \nu_1, \quad \omega_1 = n_2 t + \nu_2, \quad \omega_2 = \omega_{20} = \text{const}$$

where $n_1 = -\rho_1/A + L/C$, $n_2 = (\rho_1 - L)/A$ are the frequencies of unperturbed motion, and ν_1 and ν_2 are constants of integration.

To the perturbation function $R = \varepsilon(H_1 + H_2 + H_3)$ we now apply the averaging operator $M_{\lambda,\omega_1}[R]$ [2] with respect to the fast angular variables λ and ω_1, after which the averaged Hamiltonian can be written in the form

$$\overline{H} = \frac{\rho_1(\rho_1 - 2L)}{2A} + \frac{L^2}{2C} + \varepsilon\left[\frac{\rho_1(\rho_1 - 2L)}{4B} + \frac{\alpha_3 DL\Omega}{C} - \beta_3 mgb_2\right]$$

The corresponding canonical averaged equations have the form

$$\begin{aligned} &\frac{d\overline{L}}{dt} = 0, \quad \frac{d\overline{\rho}_1}{dt} = 0, \quad \frac{d\overline{\rho}_2}{dt} = 0 \\ &\frac{d\overline{\lambda}}{dt} = -\frac{\rho_1}{A} + \frac{L}{C} - \varepsilon\left(\frac{\rho_1}{2B} - \frac{\alpha_3 D\Omega}{C} + \beta_3 mgb_{2,L}\right) \\ &\frac{d\overline{\omega}_1}{dt} = \frac{\rho_1 - L}{A} + \varepsilon\left(\frac{L - \rho_1}{2B} + \beta_3 mgb_{2,\rho_1}\right); \quad \frac{d\overline{\omega}_2}{dt} = -\varepsilon\beta_3 mgb_{2,\rho_2} \end{aligned} \tag{2}$$

where

$$b_{2,L} = b_2\left[\frac{2L - \rho_1 - \rho_2}{L(L - \rho_1 - \rho_2)} - \frac{2}{L - \rho_1}\right], \quad b_{2,\rho_1} = b_2\left(\frac{2L}{L - \rho_1} - \frac{1}{L - \rho_1 - \rho_2}\right), \quad b_{2,\rho_2} = -\frac{L}{L - \rho_1 - \rho_2}$$

Integrating system (2), we obtain

$$\begin{aligned} &\overline{L} = \overline{L}_0 = \text{const}, \quad \overline{\lambda} = (n_1 + N_1)t + \nu_1'; \quad \overline{\rho}_1 = \overline{\rho}_{10} = \text{const}, \quad \overline{\omega}_1 = (n_2 + N_2)t + \nu_2' \\ &\overline{\rho}_2 = \overline{\rho}_{20} = \text{const}, \quad \overline{\omega}_2 = N_3 t + \nu_3 \end{aligned} \tag{3}$$

where

$$N_1 = -\varepsilon\left(\frac{\rho_1}{2B} - \frac{\alpha_3 D\Omega}{C} + \beta_3 mgb_{2,L}\right), \quad N_2 = \varepsilon\left(\frac{\rho_1 - L}{2B} - \beta_3 mgb_{2,\rho_1}\right), \quad N_3 = -\varepsilon\beta_3 mgb_{2,\rho_2}$$

(ν_1', ν_2', ν_3 are the constants of integration).

It is clear from (3) that the averaged system (2) describes pseudo-regular precession.

We can determine the type of motion of the gyrostat described by the complete system, rather than the averaged system of equations, by using Kolmogorov–Arnol'd theorem [3]. According to this theorem, perturbed motion (that is, motion described by a system of which some terms can be omitted during averaging) is conditionally periodic if the unperturbed motion is not intrinsically degenerate, that is

$$\operatorname{Hess}\overline{H} = \frac{\partial(\partial\overline{H}/\partial L,\ \partial\overline{H}/\partial\rho_1,\ \partial\overline{H}/\partial\rho_2)}{\partial(L, \rho_1, \rho_2)} \neq 0 \tag{4}$$

Computing the partial derivatives that appear in the Hessian, we obtain

$$\operatorname{Hess}\overline{H} = \frac{\beta_3^2 m^2 g^2 [C(L + \rho_1)(3L - \rho_1) - 4L^2 A]}{AC(L - \rho_1)}\varepsilon^2 + O(\varepsilon^3)$$

Thus if $C(L + \rho_1)(3L - \rho_1) \neq 4L^2 A$, inequality (4) holds, that is, the motion of the gyrostat described by the averaged system of equations cannot degenerate and thus, by the Kolmogorov–Arnol'd theorem [3], it will be stable with respect to the quantities L, ρ_1 and ρ_2. This in turn implies that the perturbed motion described by the complete system of equations will be conditionally periodic for all $t \geqslant t_0$.

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